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On the Shape Operator of Biconservative Hypersurfaces in $\mathbb{E}^5_2$

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Abstract. In this paper, we present a short survey about recent results on biconservative hypersurfaces in pseudo-Riemannian space forms and discuss some open problems on this topic. Further, we consider canonical forms of the shape operator of biconservative hypersurface of index 2 in $\mathbb{E}^5_2$ and by choosing an appropriate base field, we obtain that there are six possible canonical forms of the shape operator satisfying $\langle \nabla H, \nabla H \rangle = 0$, i.e., ∇H is light-like vector.

Keywords. Biconservative hypersurface, pseudo-Euclidean space, biharmonic submanifold, shape operator.
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1 Introduction

In 1981, Nomizu introduced isoparametric hypersurfaces in Lorentzian space forms. A hypersurface is called isoparametric if the minimal polynomial of shape operator is constant. It is well-known that the shape operator of a Riemannian submanifold is always diagonalizable, but this is not the case for the shape operator of a Lorentzian submanifold. This makes the isoparametric theory in pseudo-Riemannian space forms different from that in Riemannian space forms. In [5], Magid classified Lorentzian isoparametric hypersurfaces and obtained that the shape operator of a Lorentzian hypersurface in a Minkowski space can have four possible canonical forms by choosing an appropriate frame field. He obtained this result by Petrov's consideration in [6] , i.e., a symmetric endomorphism of a vector space with a Lorentzian inner product can be put into one of four possible canonical forms. By considering recent results obtained by Turgay in [7] and Deepika in [1], one can conclude that there is only two different families of biconservative hypersurfaces in $\mathbb{E}^4_1$ by considering the canonical forms of their shape operator (see Theorem 3.3).

Now, there arise a natural question: What will be the canonical forms of the shape operator if we increase the dimension of the pseudo-Euclidean space as well as index of the hypersurface? Thus, one can ask a general question which is still open to all, i.e., "Does there exist any specific formula from which one can get all possible canonical forms of the shape operator of a hypersurface with variable index of general ambient pseudo-Euclidean space $\mathbb{E}^n_s$" ? So it is natural to start index 2 hypersurfaces in $\mathbb{E}^5_2$. During study, it is observed that if one consider index 2 hypersurfaces in $\mathbb{E}^5_2$ then the number of canonical forms of the shape operator increases to 9 whereas it is 4 in case of $\mathbb{E}^4_1$.

The paper is organized as follows. In Sect. 2, we give some basic definitions and formulas which we used in other sections of the paper. In Sect. 3, we present a short survey about recent papers on biconservative hypersurfaces and try to point out problems which left open in these papers. In Sect. 4, we study existance of all possible canonical forms of the shape operator of biconservative hypersurfaces of index 2 with an additional condition, i.e., ∇H is a lightlike vector whereas H is mean curvature vector field of the hypersurface, and further, we obtain our main result.

2 Preliminaries

In this section we recall some basic definitions and formulas that we will use in other part of the paper.

2.1 Hypersurfaces of $\mathbb{E}^5_2$

Let $\mathbb{E}^5_2$ denote the 5-dimensional real vector space $\mathbb{R}^5$ with the canonical inner product of signature $(2,3)$ given by

$$\widetilde{g}(x,y) = \langle x,y\rangle = -x_1y_1 - x_2y_2 + x_3y_3 + x_4y_4 + x_5y_5.$$

We consider an oriented hypersurface M of $\mathbb{E}^5_2$ with index 2. Let N be its unit normal vector associated with the orientation of M. We define the shape operator S of M by the Weingarten formula

$$\widetilde{\nabla}_X N = -SX,$$

where X is a vector field tangent to M and $\widetilde{\nabla}$ denotes the Levi-Civita connection of $\mathbb{E}^5_2$. Let ∇ stands for the Levi-Civita connection of M with respect to the induced metric on M, then the Gauss formula is given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y)$$

where h is the second fundamental form of M. Note that h and S are related with the equation

$$\langle SX, Y\rangle = \langle h(X,Y), N\rangle. \tag{2.1}$$

The eigenvalues of S are called principal curvatures of M. Corresponding to every principal curvature k, we have algebraic multiplicity and geometric multiplicity. Algebraic multiplicity ν is the exponent of $(x--k)$ in the characteristic polynomial and geometric multiplicity μ is the dimension of the eigenspace

$$T_k = \{X \in T_x M : SX = kX\}.$$

A principal curvature k is called diagonalizable if $\nu = \mu$. The hypersurface M is called biconservative if it satisfies

$$S(\nabla H) = -2H\nabla H. \tag{BC}$$

Remark 2.1. If M has constant mean curvature, then (BC) is satisfied trivially. Hence, in the remaining part of the paper, we will assume that ∇H does not vanish on M, i.e., there exists a vector field X on M such that $X(H) \neq 0$.

On the other hand, since the ambient space $\mathbb{E}^5_2$ is a flat space, its curvature tensor $\tilde{R}$ vanishes identically. Thus, Gauss and Codazzi equations become

$$\left(\tilde{R}(X,Y)Z\right)^T = 0, \quad \text{and} \quad \left(\tilde{R}(X,Y)Z\right)^\perp = 0,$$

respectively, for any vector fields X, Y, Z tangent to M.

2.2 Shape operator of hypersurfaces with index 2 in $\mathbb{E}^5_2$

In this section, we consider canonical forms of the shape operator index 2 hypersurfaces in the pseudo-Euclidean space $\mathbb{E}^5_2$.

By a frame field in a hypersurface M of $\mathbb{E}^5_2$, we mean a base field $\{e_1, e_2, e_3, e_4\}$ of the tangent bundle of M. We put $\varepsilon_i = \langle e_i, e_i \rangle \in \{-1, 0, 1\}$. Furthermore, we define the connection forms ω_{ij} corresponding to a given frame field by

$$\omega_{ij}(e_k) = \langle \nabla_{e_k} e_i, e_j \rangle.$$

Now, it is well-known that shape operator S is symmetric because of (2.1). On the other hand, there is a well-known theorem that a symmetric operator in a positive definite vector space is diagonalizable over R. The situation is more complicated if the metric is not definite, and the complete answer is given by the following theorem of Petrov [6].

Theorem 2.2. ***(Principal Axis Theorem for a Tensor [6])*** *In a vector space V^n with a non- degenerate metric g every symmetric operator S can be put into the following form:*

$$S = \begin{pmatrix} B_{p_1} & & & & & & & \\ & \cdot & & & & & & \\ & & \cdot & & & & & \\ & & & \cdot & & & & \\ & & & & B_{p_k} & & & \\ & & & & & C_{2t_1} & & \\ & & & & & & \cdot & \\ & & & & & & & \cdot \\ & & & & & & & & \cdot \\ & & & & & & & & & C_{2t_m} \end{pmatrix}$$

where B_{p_i} *is* $p_i \times p_i$, C_{2t_j} *is* $2t_j \times 2t_j$ *and* $\sum p_i + 2\sum t_j = n$. *This is with respect to a basis for which the inner product has the form*

$$\begin{pmatrix} g_{p_1} & & & & & & & & \\ & \cdot & & & & & & & \\ & & \cdot & & & & & & \\ & & & \cdot & & & & & \\ & & & & g_{p_k} & & & & \\ & & & & & g_{2t_1} & & & \\ & & & & & & \cdot & & \\ & & & & & & & \cdot & \\ & & & & & & & & \cdot \\ & & & & & & & & & g_{2t_m} \end{pmatrix}$$

$$B_{p_i} = \begin{pmatrix} l_i\lambda_i & l_i & & & & & & & \\ & \cdot & & & & & & & \\ & & \cdot & \cdot & & & & & \\ & & & \cdot & \cdot & & & & \\ & & & & \cdot & & & & \\ & & & & & \cdot & \cdot & & \\ & & & & & & \cdot & \cdot & \\ & & & & & & & \cdot & l_i \\ & & & & & & & & l_i\lambda_i \end{pmatrix}$$

$$C_{2j} = \begin{pmatrix} \alpha_j & \beta_j & 1 & 0 & & & & & & \\ -\beta_j & \alpha_j & 0 & 1 & & & & & & \\ & & \alpha_j & \beta_j & 1 & 0 & & & & \\ & & -\beta_j & \alpha_j & 0 & 1 & & & & \\ & & & & \cdot & & & & & \\ & & & & & \cdot & & & & \\ & & & & & & \cdot & & & \\ & & & & & & & \cdot & & \\ & & & & & & & & \alpha_j & \beta_j \\ & & & & & & & & -\beta_j & \alpha_j \end{pmatrix}$$

and

$$g_{p_i} = \begin{pmatrix} & & & l_i \\ & & \cdot & \\ & \cdot & & \\ \cdot & & & \\ l_i & & & \end{pmatrix}, \quad g_{2t_j} = \begin{pmatrix} & & & & & 1 & 0 \\ & & & & & 0 & -1 \\ & & & & \cdot & & \\ & & & \cdot & & & \\ & & \cdot & & & & \\ 1 & 0 & & & & & \\ 0 & -1 & & & & & \end{pmatrix}.$$

Now, with the help of Theorem 2.2 and by choosing an appropriate base field $\{e_1, e_2, e_3, e_4\}$ of the tangent bundle of M, we obtain that the possible canonical forms of the shape operator S of $\mathbb{E}^5_2$ can have one of the following forms. Note that in each cases below, g denotes the induced metric tensor of M, i.e., $g_{ij} = \langle e_i, e_j \rangle$.

$$\text{Case I.}\ S = \begin{pmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & k_4 \end{pmatrix}, \qquad g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix};$$

$$\text{Case II.}\ S = \begin{pmatrix} k_1 & 1 & 0 & 0 \\ 0 & k_1 & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & k_4 \end{pmatrix}, \qquad g = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix};$$

$$\text{Case III.}\ S = \begin{pmatrix} k_1 & 1 & 0 & 0 \\ 0 & k_1 & 0 & 0 \\ 0 & 0 & k_3 & 1 \\ 0 & 0 & 0 & k_3 \end{pmatrix}, \qquad g = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix};$$

$$\text{Case IV.}\ S = \begin{pmatrix} k_1 & 1 & 0 & 0 \\ 0 & k_1 & 0 & 0 \\ 0 & 0 & k_3 & \beta_1 \\ 0 & 0 & -\beta_1 & k_3 \end{pmatrix}, \qquad g = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix};$$

$$\text{Case V.}\ S = \begin{pmatrix} k_1 & 0 & 1 & 0 \\ 0 & k_1 & 0 & 0 \\ 0 & -1 & k_1 & 0 \\ 0 & 0 & 0 & k_4 \end{pmatrix}, \qquad g = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix};$$

$$\text{Case VI.}\ S = \begin{pmatrix} k_1 & \beta_1 & 0 & 0 \\ -\beta_1 & k_1 & 0 & 0 \\ 0 & 0 & k_3 & \beta_2 \\ 0 & 0 & -\beta_2 & k_3 \end{pmatrix}, \qquad g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix};$$

$$\text{Case VII.}\ S = \begin{pmatrix} k_1 & \beta_1 & 1 & 0 \\ -\beta_1 & k_1 & 0 & 1 \\ 0 & 0 & k_1 & \beta_1 \\ 0 & 0 & -\beta_1 & k_1 \end{pmatrix}, \qquad g = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix};$$

$$\text{Case VIII.}\ S = \begin{pmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & \beta_1 \\ 0 & 0 & -\beta_1 & k_3 \end{pmatrix}, \qquad g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix};$$

$$\text{Case IX.}\ S = \begin{pmatrix} k_1 & 0 & 1 & 0 \\ 0 & k_1 & 0 & 0 \\ 0 & 0 & k_1 & 1 \\ 0 & 1 & 0 & k_1 \end{pmatrix}, \qquad g = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix};$$

for some smooth functions $k_1, k_2, k_3, k_4, \beta_1, \beta_2$.

3 Recent Results about Biconservative Hypersurfaces

3.1 Shape operator of biconservative hypersurfaces in Minkowski spaces.

The second named author obtained the following results by considering the shape operator of biconservative hypersurfaces in a Minkowski space of arbitrary dimension (See [7, Theorem 4.1]).

Theorem 3.1. *[7] Let M be a hypersurface in the Minkowski space $\mathbb{E}^4_1$, S its shape operator and H its mean curvature. Assume that ∇H is light-like and S has the minimal polynomial*

$$P(\lambda) = \prod_{i=1}^{t} (\lambda - k_1)^2 (\lambda - k_2)(\lambda - k_3) \cdots (\lambda - k_t)$$

for some t. If $t \leq 5$, then M is not biconservative.

On the other hand, in [1], Deepika considered hypersurface with complex principle curvature in an arbitrary Minkowski space and obtained the following result.

Theorem 3.2. *[1] Let M^n_1 in $\mathbb{E}^{n+1}_1$ be a biconservative Lorentz hypersurface having non diagonal shape operator with complex eigenvalues and with at most five distinct principal curvatures. Then M^n_1 has constant mean curvature.*

By combining these results, we would like to state the following result on the shape operator of biconservative hypersurface in $\mathbb{E}^4_1$.

Theorem 3.3. *Let M be a hypersurface in $\mathbb{E}^4_1$ and H its mean curvature. Then by choosing an appropriated frame field $\{e_1, e_2, e_3\}$ the matrix representation of the shape operator S of M can have one of the following two canonical forms*

$$\begin{aligned} &\text{Case 1.} \quad S = \begin{pmatrix} -\frac{3\varepsilon}{2}H & & \\ & k_2 & \\ & & \frac{3}{2}(2+\varepsilon)H - k_2 \end{pmatrix} \quad \text{for a function } k_2, \\ &\text{Case 2.} \quad S = \begin{pmatrix} -\frac{3H}{2} & & \\ & \frac{9H}{4} & 1 \\ & & \frac{9H}{4} \end{pmatrix}, \end{aligned} \tag{3.1}$$

where e_1 is proportional to ∇H and ε is the signature of the normal of M, i.e.,

$$\varepsilon = \begin{cases} -1 & \text{if } M \text{ is Riemannian} \\ 1 & \text{if } M \text{ is Lorentzian.} \end{cases}$$

At this instant, we would like to mention that the complete classification of biconservative hypersurfaces, given in Case 1 of (3.1), is obtained by Yu Fu and the second named author in [4] (See Sect. 3.2). However, the following problem is still open.

Problem 1. Classify all biconservative hypersurfaces in $\mathbb{E}^4_1$ with the shape operator given in the Case 2 of (3.1).

3.2 Biconservative hypersurfaces in Minkowski spaces.

In [2], Yu Fu obtained the following results.

Proposition 3.4. *[2] Let M be a nondegenerate biconservative surface immersed in the 3-dimensional Minkowski space $\mathbb{E}^3_1$. Then the immersed surface M is either a CMC surface or locally given by one of the following eight surfaces.*

1. *A timelike surface of revolution with spacelike axis, given by*

$$x(s,t) = (f(s), s\cosh t, s\sinh t) \tag{3.2}$$

where $s \in (27, +\infty)$ and

$$f(s) = \frac{9}{2}(s^{\frac{1}{3}}\sqrt{s^{\frac{2}{3}} - 9} + 9In(s^{\frac{1}{3}} + \sqrt{s^{\frac{2}{3}} - 9})).$$

2. *A spacelike surface of revolution with spacelike axis, given by*

$$x(s,t) = (f(s), s\sinh t, s\cosh t) \tag{3.3}$$

where $s \in (0, 27)$ and

$$f(s) = \frac{81}{2}arcsin\frac{1}{3}s^{\frac{1}{3}} - \frac{9}{2}s^{\frac{1}{3}}\sqrt{9 - s^{\frac{2}{3}}}.$$

3. *A spacelike surface of revolution with timelike axis, given by*

$$x(s,t) = (s\cos t, s\sin t, f(s)), \tag{3.4}$$

where $s \in (0, +\infty)$ and

$$f(s) = \frac{9}{2}(s^{\frac{1}{3}}\sqrt{s^{\frac{2}{3}} + 9} - 9In(s^{\frac{1}{3}} + \sqrt{s^{\frac{2}{3}} + 9})).$$

4. *A spacelike surface of revolution with lightlike axis, given by*

$$x(s,t) = (\frac{1}{2}st^2 - \frac{1}{30}s^{\frac{5}{3}} - \frac{1}{2}s, st, \frac{1}{2}st^2 - \frac{1}{30}s^{\frac{5}{3}} + \frac{1}{2}s), \tag{3.5}$$

where $s \in (0, +\infty)$.

5. *A timelike surface of revolution with spacelike axis, given by*

$$x(s,t) = (f(s), s\sinh t, s\cosh t), \tag{3.6}$$

where $s \in (0, +\infty)$ *and*

$$f(s) = \frac{9}{2}(s^{\frac{1}{3}}\sqrt{s^{\frac{2}{3}}+9} - 9In(s^{\frac{1}{3}} + \sqrt{s^{\frac{2}{3}}+9})).$$

6. *A timelike surface of revolution with timelike axis, given by*

$$x(s,t) = (s\cos t, s\sin t, f(s)), \tag{3.7}$$

where $s \in (0, 27)$ *and*

$$f(s) = \frac{81}{2}arcsin\frac{1}{3}s^{\frac{1}{3}} - \frac{9}{2}s^{\frac{1}{3}}\sqrt{9 - s^{\frac{2}{3}}}.$$

7. *A timelike surface of revolution with lightlike axis, given by*

$$x(s,t) = (\frac{1}{2}st^2 + \frac{1}{30}s^{\frac{5}{3}} - \frac{1}{2}s, st, \frac{1}{2}st^2 + \frac{1}{30}s^{\frac{5}{3}} + \frac{1}{2}s), \tag{3.8}$$

where $s \in (0, +\infty)$.

8. *A null sroll with non-constant mean curvature.*

In [3], Yu Fu give a complete explicit classification of biconservative surfaces in de Sitter 3-spaces and anti-de Sitter 3-spaces. He obtained the following results.

Proposition 3.5. *[3] Let M be a nondegenerate bi-conservative surface immersed in the 3-dimensional de Sitter space $\mathbb{S}^3_1(1) \in \mathbb{E}^4_1$. Then the immersed surface M is either a CMC surface or locally given by one of the following nine surfaces.*

1. *A timelike rotational surface, given by*

$$x(s,t) = (s\sinh t, s\cosh t, \sqrt{1-s^2}\cos f, \sqrt{1-s^2}\sin f), \tag{3.9}$$

where $s \in (0, 1)$ *and*

$$f(s) = \pm\int \frac{3s^{-\frac{1}{3}}}{(1-s^2)\sqrt{1-9s^{-\frac{2}{3}}-s^2}}ds.$$

2. *A spacelike rotational surface, given by*

$$x(s,t) = (s\cosh t, s\sinh t, \sqrt{1+s^2}cosf, \sqrt{1+s^2}sinf), \tag{3.10}$$

where $s \in (0, +\infty)$ *and*

$$f(s) = \pm\int \frac{3s^{-\frac{1}{3}}}{(1+s^2)\sqrt{9s^{-\frac{2}{3}}-s^2-1}}ds.$$

3. A spacelike rotational surface, given by

$$x(s,t) = (\sqrt{1-s^2}\sinh f, \sqrt{1-s^2}\cosh f, scost, ssint), \tag{3.11}$$

where $s \in (0,1)$ *and*

$$f(s) = \pm \int \frac{3s^{-\frac{1}{3}}}{(1-s^2)\sqrt{1+9s^{-\frac{2}{3}}-s^2}} ds.$$

4. A spacelike rotational surface, given by

$$x(s,t) = (\sqrt{s^2-1}\cosh f, \sqrt{s^2-1}\sinh f, scost, ssint), \tag{3.12}$$

where $s \in (1,+\infty)$ *and*

$$f(s) = \pm \int \frac{3s^{-\frac{1}{3}}}{(s^2-1)\sqrt{1+9s^{-\frac{2}{3}}-s^2}} ds.$$

5. A spacelike rotational surface, given by

$$x(s,t) = (\frac{1}{2}(st^2+sf^2-\frac{1}{s}+s), \frac{1}{2}(st^2+sf^2-\frac{1}{s}-s), sf, st) \tag{3.13}$$

where $s \in (0, 3^{\frac{3}{4}})$ *and*

$$f(s) = \int \frac{3}{s^2\sqrt{9-s^{-\frac{8}{3}}}} ds.$$

6. A timelike rotational surface, given by

$$x(s,t) = (s\cosh t, s\sinh t, \sqrt{1+s^2}cosf, \sqrt{1+s^2}sinf), \tag{3.14}$$

where $s \in (0,+\infty)$ *and*

$$f(s) = \pm \int \frac{3s^{-\frac{1}{3}}}{(1+s^2)\sqrt{9s^{-\frac{2}{3}}+s^2+1}} ds.$$

7. A timelike rotational surface, given by

$$x(s,t) = (\sqrt{1-s^2}\sinh f, \sqrt{1-s^2}\cosh f, scost, ssint), \tag{3.15}$$

where $s \in (0,1)$ *and*

$$f(s) = \pm \int \frac{3s^{-\frac{1}{3}}}{(1-s^2)\sqrt{9s^{-\frac{2}{3}}+s^2-1}} ds.$$

8. *A timelike rotational surface, given by*

$$x(s,t) = (\sqrt{s^2-1}\cosh f, \sqrt{s^2-1}\sinh f, scost, ssint), \tag{3.16}$$

where $s \in (1, +\infty)$ *and*

$$f(s) = \pm \int \frac{3s^{-\frac{1}{3}}}{(s^2-1)\sqrt{9s^{-\frac{2}{3}} + s^2 - 1}} ds.$$

9. *A timelike rotational surface, given by*

$$x(s,t) = (\frac{1}{2}(st^2 + sf^2 - \frac{1}{s} + s), \frac{1}{2}(st^2 + sf^2 - \frac{1}{s} - s), sf, st) \tag{3.17}$$

where $s \in (0, +\infty)$ *and*

$$f(s) = \int \frac{3}{s^2\sqrt{9 + s^{-\frac{8}{3}}}} ds.$$

Proposition 3.6. *[3] Let* M *be a nondegenerate bi-conservative surface immersed in the 3-dimensional anti-de Sitter space* $\mathbb{H}^3_1(-1) \in \mathbb{E}^4_2$*. Then the immersed surface* M *is either a CMC surface or locally given by one of the following eleven surfaces.*

1. *A timelike rotational surface, given by*

$$x(s,t) = (s\sinh t, \sqrt{1+s^2}\cosh f, s\cosh t, \sqrt{1+s^2}\sinh f), \tag{3.18}$$

where $s \in (0, +\infty)$ *and*

$$f(s) = \pm \int \frac{3s^{-\frac{1}{3}}}{(1+s^2)\sqrt{1 - 9s^{-\frac{2}{3}} + s^2}} ds.$$

2. *A spacelike rotational surface, given by*

$$x(s,t) = (s\cosh t, \sqrt{1-s^2}\cosh f, s\sinh t, \sqrt{1-s^2}\sinh f), \tag{3.19}$$

where $s \in (0, 1)$ *and*

$$f(s) = \pm \int \frac{3s^{-\frac{1}{3}}}{(1-s^2)\sqrt{9s^{-\frac{2}{3}} + s^2 - 1}} ds.$$

3. *A spacelike rotational surface, given by*

$$x(s,t) = (s\cosh t, \sqrt{s^2-1}\sinh f, s\sinh t, \sqrt{s^2-1}\cosh f), \tag{3.20}$$

where $s \in (1, +\infty)$ *and*

$$f(s) = \pm \int \frac{3s^{-\frac{1}{3}}}{(s^2-1)\sqrt{9s^{-\frac{2}{3}} + s^2 - 1}} ds.$$

4. A spacelike rotational surface, given by

$$x(s,t) = (\sqrt{1+s^2}cosf, \sqrt{1+s^2}sinf, scost, ssint), \tag{3.21}$$

where $s \in (0,+\infty)$ *and*

$$f(s) = \pm \int \frac{3s^{-\frac{1}{3}}}{(1+s^2)\sqrt{1+9s^{-\frac{2}{3}}-s^2}}ds.$$

5. A spacelike rotational surface, given by

$$x(s,t) = (scost, ssint, \sqrt{s^2-1}cosf, \sqrt{s^2-1}sinf), \tag{3.22}$$

where $s \in (1,+\infty)$ *and*

$$f(s) = \pm \int \frac{3s^{-\frac{1}{3}}}{(s^2-1)\sqrt{s^2-9s^{-\frac{2}{3}}-1}}ds.$$

6. A spacelike rotational surface, given by

$$x(s,t) = (\frac{1}{2}(st^2 - sf^2 + \frac{1}{s} + s), sf, \frac{1}{2}(st^2 - sf^2 + \frac{1}{s} - s), st) \tag{3.23}$$

where $s \in (0,+\infty)$ *and*

$$f(s) = \int \frac{3}{s^2\sqrt{9+s^{\frac{8}{3}}}}ds.$$

7. A timelike rotational surface, given by

$$x(s,t) = (\frac{1}{2}(st^2 + sf^2 + \frac{1}{s} + s), sf, \frac{1}{2}(st^2 + sf^2 + \frac{1}{s} - s), sf), \tag{3.24}$$

where $s \in (3^{\frac{3}{4}},+\infty)$ *and*

$$f(s) = \int \frac{3}{s^2\sqrt{s^{\frac{8}{3}}-9}}ds.$$

8. A timelike rotational surface, given by

$$x(s,t) = (s\cosh t, \sqrt{1-s^2}\cosh f, s\sinh t, \sqrt{1-s^2}\sinh f), \tag{3.25}$$

where $s \in (0,1)$ *and*

$$f(s) = \pm \int \frac{3s^{-\frac{1}{3}}}{(1-s^2)\sqrt{9s^{-\frac{2}{3}}-s^2+1}}ds.$$

9. A timelike rotational surface, given by

$$x(s,t) = (s\cosh t, \sqrt{s^2-1}\sinh f, s\sinh t, \sqrt{s^2-1}\cosh f), \tag{3.26}$$

where $s \in (1, +\infty)$ *and*

$$f(s) = \pm \int \frac{3s^{-\frac{1}{3}}}{(s^2-1)\sqrt{9s^{-\frac{2}{3}} - s^2 + 1}} ds.$$

10. A timelike rotational surface, given by

$$x(s,t) = (\sqrt{1+s^2}cosf, \sqrt{1+s^2}sinf, scost, ssint), \tag{3.27}$$

where $s \in (0, +\infty)$ *and*

$$f(s) = \pm \int \frac{3s^{-\frac{1}{3}}}{(1+s^2)\sqrt{9s^{-\frac{2}{3}} - s^2 - 1}} ds.$$

11. A timelike rotational surface, given by

$$x(s,t) = (\frac{1}{2}(st^2 - sf^2 + \frac{1}{s} + s), sf, \frac{1}{2}(st^2 - sf^2 + \frac{1}{s} - s), st) \tag{3.28}$$

where $s \in (0, 3^{\frac{3}{4}})$ *and*

$$f(s) = \int \frac{3}{s^2\sqrt{9 - s^{\frac{8}{3}}}} ds.$$

Further, in [4], the author and Yu Fu considered biconservative hypersurfaces in the Minkowski 4-space with diagonalizable shape operator. They obtained the following results.

Proposition 3.7. *[4] Let* M *be a hypersurface in* $\mathbb{E}^4_1$ *given by*

$$\begin{aligned} x(s,t,u) = \bigg(& \frac{1}{2}s(t^2+u^2) + au^2 + s + \phi(s), st, (s+2a)u, \\ & \frac{1}{2}s(t^2+u^2) + au^2 + \phi(s) \bigg), \quad a \neq 0. \end{aligned} \tag{3.29}$$

Then, M *is biconservative if and only if either* M *is Riemannian and*

$$\phi(s) = c_1 \left(\ln(s+2a) - \ln s - \frac{a}{s} - \frac{a}{s+2a} \right) - \frac{s}{2}$$

or it is Lorentzian and

$$\phi(s) = c_1 \int_{s_0}^{s} \left(\xi(\xi+2a)\right)^{2/3} d\xi - \frac{s}{2},$$

where $c_1 \neq 0$ *and* s_0 *are some constants.*

Theorem 3.8. *[4] Let M be a hypersurface in $\mathbb{E}^4_1$ with diagonalizable shape operator and three distinct principal curvatures. Then M is biconservative if and only if it is congruent to one of hypersurfaces*

1. *A generalized cylinder $M^2_0 \times \mathbb{E}^1_1$ where M is a biconservative surface in $\mathbb{E}^3$;*
2. *A generalized cylinder $M^2_0 \times \mathbb{E}^1$ where M is a biconservative Riemannian surface in $\mathbb{E}^3_1$;*
3. *A generalized cylinder $M^2_1 \times \mathbb{E}^1$, where M is a biconservative Lorentzian surface in $\mathbb{E}^3_1$;*
4. *A Rimannian surface given by*
$$x(s,t,u) = (s\,\cosh t, s\,\sinh t, f_1(s)\cos u, f_1(s)\sin u) \tag{3.30}$$
for a function f_1 satisfying
$$\frac{f_1''}{f_1'^2 - 1} = \frac{f_1 f_1' + s}{s f_1};$$
5. *A Lorentzian surface with the parametrization given in* (3.30) *for a function f_1 satisfying*
$$\frac{-3f_1''}{f_1'^2 - 1} = \frac{f_1 f_1' + s}{s f_1};$$
6. *A Rimannian surface given by*
$$x(s,t,u) = (s\,\sinh t, s\,\cosh t, f_2(s)\cos u, f_2(s)\sin u) \tag{3.31}$$
for a function f_2 satisfying
$$\frac{f_2''}{f_2'^2 + 1} = \frac{f_2 f_2' + s}{s f_2};$$
7. *A surface given in Proposition 3.7.*

3.3 Biconservative Hypersurfaces in $\mathbb{E}^5_2$

In [8], we study biconservative hypersurfaces of index 2 in $\mathbb{E}^5_2$ and obtain the complete classification of biconservative hypersurfaces with diagonalizable shape operator at exactly three distinct principal curvatures. The results are following.

Theorem 3.9. *[8] Let M be an oriented biconservative hypersurface of index 2 in the pseudo-Euclidean space $\mathbb{E}^5_2$. Assume that its shape operator has the form*
$$S = \mathrm{diag}(k_1, 0, 0, k_4), \quad k_4 \neq 0.$$
Then, it is congruent to one of the following eight type of generalized cylinders over surfaces for some smooth functions $\phi = \phi(s)$ and $\psi = \psi(s)$.

(i). $x(s,t,u,v) = (t, u, \phi\cos v, \phi\sin v, \psi), \quad \phi'^2 + \psi'^2 = 1;$

(ii). $x(s,t,u,v) = (\phi\sinh v, t, u, \phi\cosh v, \psi), \quad \phi'^2 + \psi'^2 = 1;$

(iii). $x(s,t,u,v) = (\psi, t, u, \phi\cos v, \phi\sin v), \quad \phi'^2 - \psi'^2 = -1;$

(iv). $x(s,t,u,v) = (\phi\cosh v, t, u, \phi\sinh v, \psi), \quad \phi'^2 - \psi'^2 = 1;$

(v). $x(s,t,u,v) = \left(\frac{v^2 s}{2} + \psi + s, t, u, vs, \frac{v^2 s}{2} + \psi\right), \ 1 - 2\psi' < 0;$

(vi). $x(s,t,u,v) = (\phi\cos v, \phi\sin v, t, u, \psi), \quad \phi'^2 - \psi'^2 = 1;$

(vii). $x(s,t,u,v) = (\phi\sinh v, \psi, t, u, \phi\cosh v), \quad \phi'^2 - \psi'^2 = -1;$

(viii). $x(s,t,u,v) = \left(\frac{sv^2}{2} + \psi, sv, t, u, \frac{sv^2}{2} + \psi + s\right), \ 1 + 2\psi' < 0.$

Theorem 3.10. *[8] Let M be an oriented hypersurface of index 2 in the pseudo-Euclidean space $\mathbb{E}^5_2$. Assume that its shape operator has the form*

$$S = \mathrm{diag}(k_1, k_2, k_2, 0), \quad k_2 \neq 0.$$

Then, it is congruent to one of the following eight type of cylinders for some smooth functions $\phi = \phi(s)$ and $\psi = \psi(s)$.

(i). $x(s,t,u,v) = (v, \phi\cosh t, \phi\sinh t\cos u, \phi\sinh t\sin u, \psi), \quad \phi'^2 - \psi'^2 = 1;$

(ii). $x(s,t,u,v) = (v, \psi, \phi\cos t, \phi\sin t\cos u, \phi\sin t\sin u), \quad \phi'^2 - \psi'^2 = -1;$

(iii). $x(s,t,u,v) = (\phi\cosh t\sin u, \phi\cosh t\cos u, \phi\sinh t, \psi, v), \quad \phi'^2 - \psi'^2 = 1;$

(iv). $x(s,t,u,v) = (\psi, \phi\sinh t, \phi\cosh t\cos u, \phi\cosh t\sin u, v), \quad \phi'^2 - \psi'^2 = -1;$

(v). $x(s,t,u,v) = (v, \phi\sinh t, \phi\cosh t\cos u, \phi\cosh t\sin u, \psi), \quad \phi'^2 + \psi'^2 = 1;$

(vi). $x(s,t,u,v) = (\phi\sinh t cosu, \phi\sinh t\sin u, \phi\cosh u, \psi, v), \quad \phi'^2 + \psi'^2 = 1;$

(vii). $x(s,t,u,v) = \left(\frac{s(t^2+u^2)}{2} + \psi, v, st, su, \frac{s(t^2+u^2)}{2} + \psi - s\right), \quad 1 - 2\psi' < 0;$

(viii). $x(s,t,u,v) = \left(\frac{s(t^2-u^2)}{2} + \psi, st, su, v, \frac{s(t^2-u^2)}{2} + \psi + s\right), \quad 1 + 2\psi' < 0.$

Theorem 3.11. *[8] Let M be an oriented hypersurface of index 2 in the pseudo-Euclidean space $\mathbb{E}^5_2$. Assume that its shape operator has the form*

$$S = \mathrm{diag}(k_1, k_2, k_2, k_4), \quad k_4 \neq k_2$$

for some non-vanishing smooth functions k_1, k_2, k_4. Then, it is congruent to one of the following eight type of hypersurfaces for some smooth functions $\phi_1 = \phi_1(s)$ and $\phi_2 = \phi_2(s)$.

(i). $x(s,t,u,v) = (\phi_2 \sinh v, \phi_1 \cosh t, \phi_1 \sinh t \cos u, \phi_1 \sinh t \sin u, \phi_2 \cosh v)\,, \quad \phi_1'^2 - \phi_2'^2 = 1$;

(ii). $x(s,t,u,v) = (\phi_2 \cos v, \phi_2 \sin v, \phi_1 \cos t, \phi_1 \sin t \cos u, \phi_1 \sin t \sin u)\,, \quad \phi_1'^2 - \phi_2'^2 = -1$;

(iii). $x(s,t,u,v) = (\phi_1 \cosh t \sin u, \phi_1 \cosh t \cos u, \phi_1 \sinh t, \phi_2 \cos v, \phi_2 \sin v)\,, \quad \phi_1'^2 - \phi_2'^2 = 1$;

(iv). $x(s,t,u,v) = (\phi_2 \sinh v, \phi_1 \sinh t, \phi_1 \cosh t \cos u, \phi_1 \cosh t \sin u, \phi_2 \cosh v)\,, \quad \phi_1'^2 + \phi_2'^2 = 1$;

(v). $x(s,t,u,v) = (\phi_2 \cosh v, \phi_1 \sinh t, \phi_1 \cosh t \cos u, \phi_1 \cosh t \sin u, \phi_2 \sinh v)\,, \quad \phi_1'^2 - \phi_2'^2 = -1$;

(vi). $x(s,t,u,v) = (\phi_1 \sinh t cos u, \phi_1 \sinh t \sin u, \phi_1 \cosh u, \phi_2 \cos v, \phi_2 \sin v)\,, \quad \phi_1'^2 + \phi_2'^2 = 1$;

(vii). A hypersurface given by

$$x(s,t,u,v) = \Big(\frac{s}{2}\left(t^2+u^2-v^2\right) - av^2 + \psi, v(2a+s), st, su, \frac{s}{2}\left(t^2+u^2-v^2\right) - av^2 + \psi - s\Big) \tag{3.32}$$

for a non-zero constants a and a smooth function $\psi = \psi(s)$ such that $1 - 2\psi' < 0$;

(viii). A hypersurface given by

$$x(s,t,u,v) = \left(\frac{s\left(t^2-u^2-v^2\right)}{2} + av^2 + \psi, st, su, v(s-2a), \frac{s\left(t^2-u^2-v^2\right)}{2} + av^2 + \psi + s\right) \tag{3.33}$$

for a non-zero constants a and a smooth function $\psi = \psi(\tilde{s})$ such that $1 + 2\psi' < 0$.

4 Shape operator of biconservative hypersurfaces of index 2 in $\mathbb{E}^5_2$

In this section, we only consider hypersurfaces with non-constant mean curvature. Before we proceed, we would like to mention that in [8], authors considered hypersurfaces of index 2 in $\mathbb{E}^5_2$. It is proved that if ∇H is assumed not to be a light-like vector, then the shape operator of a biconservative hypersurface has one of the four possible canonical forms given below.

Lemma 4.1. *[8] Let M be a hypersurface of index 2 in $\mathbb{E}^5_2$ with H as its (first) mean curvature. Assume that ∇H is not light-like. If M is biconservative, then with respect to a suitable frame field $\{e_1 = \frac{\nabla H}{\|\nabla H\|}, e_2, e_3, e_4\}$, its shape operator S has one of the following forms:*

$$\begin{aligned}
&\textit{Case I. } S = \begin{pmatrix} -2H & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & k_4 \end{pmatrix}, \\
&\textit{Case II. } S = \begin{pmatrix} -2H & 0 & 0 & 0 \\ 0 & k_2 & 1 & 0 \\ 0 & 0 & k_2 & 0 \\ 0 & 0 & 0 & k_4 \end{pmatrix}, \\
&\textit{Case III. } S = \begin{pmatrix} -2H & 0 & 0 & 0 \\ 0 & k_2 & -\nu & 0 \\ 0 & \nu & k_2 & 0 \\ 0 & 0 & 0 & k_4 \end{pmatrix}, \\
&\textit{Case IV. } S = \begin{pmatrix} -2H & 0 & 0 & 0 \\ 0 & 2H & 0 & 0 \\ 0 & 0 & 2H & -1 \\ 0 & 1 & 0 & 2H \end{pmatrix},
\end{aligned} \tag{4.1}$$

for some smooth functions k_2, k_3, k_4, ν. In Cases I and III, the induced metric $g_{ij} = g(e_i, e_j) = \langle e_i, e_j \rangle$ of M is $g_{ij} = \varepsilon_i \delta_{ij} \in \{-1, 0, 1\}$, while in Cases II and IV, it is given by

$$g = \begin{pmatrix} \varepsilon_1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -\varepsilon_1 \end{pmatrix}.$$

4.1 Main Results

In this subsection, we consider the shape operator of a biconservative hypersurface in $\mathbb{E}^5_2$ with an additional hypothesis of being light-like of gradient of its mean curvature. Our aim is to investigate possible canonical forms of the shape operator S of M under the following assumption.

Assumption. ∇H is light-like, where H is the mean curvature of the biconservative hypersurface M with index 2.

By the above assumption, (BC) implies that ∇H is an eigenvector of S with corresponding eigenvalue $-2H$. It is very easy to observe that the matrix representation of S with respect to a suitable frame field $\{e_1, e_2, e_3, e_4\}$ can not be one of Case VI, Case VII or Case IX given in Sect. 2.2.

First, we obtain the following result.

Proposition 4.2. *The subspace* $\ker(S - 2HI)$ *is degenerate, where I is the identity operator acting on the space of tangent vector fields of M.*

Proof. Let us consider, the subspace $\ker(S-2HI)$ is non-degenerate. Since $\nabla H \in \Omega = \ker(S-2H)$ and it is light-like then the index of Ω should be at least 1. Thus, there exists two unit vector fields X, Y such that $SX = -2HX$, $SY = -2HY$, $\nabla H = \tau(X-Y)$ for a smooth function τ and $\langle X, X\rangle = -\langle Y, Y\rangle = 1$. Furthermore, we have $X(H) \neq 0$ and $Y(H) \neq 0$. However, this contradicts with the Codazzi equation $\left(\widetilde{R}(X,Y)X\right)^{\perp} = 0$ which yields $X(H) = 0$. □

By using this result, we conclude that the matrix representation of S with respect to a suitable frame field $\{e_1, e_2, e_3, e_4\}$ can not be one of Case I, Case VIII or Case II with $k_3 = k_4 = -2H$. Hence, we have the following result.

Lemma 4.3. *The matrix representation of S with respect to a suitable frame field $\{e_1, e_2, e_3, e_4\}$ is one of the following four forms, where we assume e_1 to be proportional to ∇H and g denotes the induced metric tensor of M, i.e., $g_{ij} = \langle e_i, e_j\rangle$.*

$$\text{Case I. } S = \begin{pmatrix} -2H & 1 & 0 & 0 \\ 0 & -2H & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & 8H-k_3 \end{pmatrix}, \qquad g = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

for a smooth function k_3;

$$\text{Case II. } S = \begin{pmatrix} -2H & 1 & 0 & 0 \\ 0 & -2H & 0 & 0 \\ 0 & 0 & 4H & 1 \\ 0 & 0 & 0 & 4H \end{pmatrix}, \qquad g = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix};$$

$$\text{Case III. } S = \begin{pmatrix} -2H & 1 & 0 & 0 \\ 0 & -2H & 0 & 0 \\ 0 & 0 & 4H & \beta_1 \\ 0 & 0 & -\beta_1 & 4H \end{pmatrix}, \qquad g = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

for a smooth function β_1;

$$\text{Case IV. } S = \begin{pmatrix} -2H & 0 & 1 & 0 \\ 0 & -2H & 0 & 0 \\ 0 & -1 & -2H & 0 \\ 0 & 0 & 0 & 10H \end{pmatrix}, \qquad g = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Now, since e_1 is proportional to ∇H, we have

$$e_1(H) = e_3(H) = e_4(H) = 0. \tag{4.2}$$

Proposition 4.4. *There exists no hypersurfaces of index 2 in $\mathbb{E}^5_2$ with shape operator given by Case II of Lemma 4.3.*

Proof. Assume that the shape operator of M is as given in Case II of Lemma 4.3. Then, the second fundamental form of M satisfies

$$h(e_1,e_2) = 2HN, \quad h(e_2,e_2) = -N, \qquad h(e_3,e_4) = -4HN, \quad h(e_4,e_4) = -N$$

and for all other cases, we have $h(e_i, e_j) = 0$.

Note that we have

$$\begin{aligned}\nabla_{e_k} e_1 =& - \omega_{12}(e_k)e_1 - \omega_{14}(e_k)e_3 - \omega_{13}(e_k)e_4,\\ \nabla_{e_k} e_2 =&\omega_{12}(e_k)e_2 - \omega_{24}(e_k)e_3 - \omega_{23}(e_k)e_4,\\ \nabla_{e_k} e_3 =&\omega_{23}(e_k)e_1 + \omega_{13}(e_k)e_2 - \omega_{34}(e_k)e_3,\\ \nabla_{e_k} e_4 =&\omega_{24}(e_k)e_1 + \omega_{14}(e_k)e_2 + \omega_{34}(e_k)e_4.\end{aligned} \tag{4.3}$$

Moreover, because of (4.2), we have $[e_1, e_3](H) = [e_1, e_4](H) = [e_3, e_4](H) = 0$ which give

$$\omega_{13}(e_1) = \omega_{14}(e_1) = 0, \qquad \omega_{14}(e_3) = \omega_{13}(e_4). \tag{4.4}$$

We apply the Codazzi equation $\left(\widetilde{R}(e_i, e_j)e_k\right)^{\perp} = 0$ for each triplet (i, j, k) in the set $\{(3, 1, 2), (3, 2, 1), (4, 1, 2), (4, 2, 1), (1, 4, 3), (1, 3, 4), (3, 2, 3), (4, 3, 4)\}$ and combine equations obtained with (4.4) and (4.3) to get

$$\begin{aligned}\omega_{23}(e_1) = \omega_{13}(e_2) = \omega_{24}(e_1) = \omega_{14}(e_2) =&0,\\ \omega_{34}(e_3) = \omega_{13}(e_4) = \omega_{13}(e_3) = \omega_{23}(e_3) =&0.\end{aligned}$$

Therefore, from (4.3) we have

$$\begin{aligned}\omega_{13} = 0, \qquad &\nabla_{e_4} e_1 = -\omega_{12}(e_4)e_1 - \omega_{14}(e_4)e_3, \quad \nabla_{e_2} e_1 = -\omega_{12}(e_2)e_1,\\ &\nabla_{e_2} e_3 = \omega_{23}(e_2)e_1 - \omega_{34}(e_2)e_3.\end{aligned} \tag{4.5}$$

However, the Gauss equation $\left(\tilde{R}(e_2, e_4)e_1\right)^{T} = 0$ implies $H \equiv 0$ on M which yields a contradiction. □

Similarly, we have

Proposition 4.5. *There exists no hypersurfaces of index 2 in $\mathbb{E}^5_2$ with shape operator given by Case IV of Lemma 4.3.*

Proof. Assume that the shape operator of M is as given in Case III of Lemma 4.3. Then, the second fundamental form of M satisfies

$$h(e_1, e_2) = 2HN, \quad h(e_2, e_3) = -N, \qquad h(e_3, e_3) = -2HN, \quad h(e_4, e_4) = -10HN$$

and for all other cases, we have $h(e_i, e_j) = 0$. Similar to proof of Proposition 4.4 we have (4.4).

Note that we have

$$\begin{aligned}\nabla_{e_k} e_1 =& - \omega_{12}(e_k)e_1 + \omega_{13}(e_k)e_3 - \omega_{14}(e_k)e_4,\\ \nabla_{e_k} e_2 =&\omega_{12}(e_k)e_2 + \omega_{23}(e_k)e_3 - \omega_{24}(e_k)e_4,\\ \nabla_{e_k} e_3 =&\omega_{23}(e_k)e_1 + \omega_{13}(e_k)e_2 - \omega_{34}(e_k)e_4,\\ \nabla_{e_k} e_4 =&\omega_{24}(e_k)e_1 + \omega_{14}(e_k)e_2 - \omega_{34}(e_k)e_3.\end{aligned} \tag{4.6}$$

We apply the Codazzi equation $\left(\widetilde{R}(e_i, e_j)e_k\right)^{\perp} = 0$ for each triplet (i, j, k) in the set $\{(3,4,3)$, $(4,1,4)$, $(4,3,4)$, $(1,2,3)$, $(1,3,2)$, $(1,2,4)$, $(1,4,2)$, $(1,3,4)$, $(1,4,3)\}$ and combine equations obtained with (4.4) and (4.6) to get

$$\omega_{12}(e_1) = \omega_{13}(e_3) = \omega_{34}(e_1) = \omega_{13}(e_4) = \omega_{34}(e_3) = 0,$$
$$\omega_{14}(e_4) = \omega_{34}(e_4) = \omega_{14}(e_2) = \omega_{24}(e_1) = 0.$$

By combining these equations with (4.6), we obtain

$$\begin{aligned}
\nabla_{e_1}e_1 = 0, \quad \nabla_{e_2}e_1 = \omega_{13}(e_2)e_3 - \omega_{12}(e_2)e_1, \quad & \nabla_{e_i}e_1 = -\omega_{12}(e_i)e_1, \\
\nabla_{e_1}e_2 = \omega_{23}(e_1)e_3, \quad & \nabla_{e_j}e_2 = \omega_{12}(e_j)e_2 + \omega_{23}(e_j)e_3 - \omega_{24}(e_j)e_4, \\
\nabla_{e_k}e_3 = \omega_{23}(e_k)e_1, \quad & \nabla_{e_2}e_3 = \omega_{23}(e_2)e_1 + \omega_{13}(e_2)e_2 - \omega_{34}(e_2)e_4, \\
\nabla_{e_1}e_4 = 0, \nabla_{e_2}e_4 = \omega_{24}(e_2)e_1 - \omega_{34}(e_2)e_3, \quad & \nabla_{e_i}e_4 = \omega_{24}(e_i)e_1.
\end{aligned}$$

for $i = 3, 4$, $j = 2, 3, 4$ and $k = 1, 3, 4$.

However, the Gauss equations $R(e_3, e_4, e_4, e_3) = 20H^2$ implies $H = 0$ on M which yields a contradiction. □

Thus, by combining Lemma 4.1, Lemma 4.3, Proposition 4.4 and Proposition 4.5, we obtain the following result.

Theorem 4.6. *Let M be a hypersurface of index 2 in $\mathbb{E}^5_2$ with H as its (first) mean curvature. If M is biconservative and ∇H is a lightlike vector, then with respect to a suitable frame field $\{e_1, e_2, e_3, e_4\}$, its shape operator S has one of* *the following six forms, where e_1 is proportional to ∇H*

$$\begin{aligned}
&\textit{Case I. } S = \begin{pmatrix} -2H & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & k_4 \end{pmatrix}, \\
&\textit{Case II. } S = \begin{pmatrix} -2H & 0 & 0 & 0 \\ 0 & k_2 & 1 & 0 \\ 0 & 0 & k_2 & 0 \\ 0 & 0 & 0 & k_4 \end{pmatrix}, \\
&\textit{Case III. } S = \begin{pmatrix} -2H & 0 & 0 & 0 \\ 0 & k_2 & -\nu & 0 \\ 0 & \nu & k_2 & 0 \\ 0 & 0 & 0 & k_4 \end{pmatrix}, \\
&\textit{Case IV. } S = \begin{pmatrix} -2H & 0 & 0 & 0 \\ 0 & 2H & 0 & 0 \\ 0 & 0 & 2H & -1 \\ 0 & 1 & 0 & 2H \end{pmatrix}, \\
&\textit{Case V. } S = \begin{pmatrix} -2H & 1 & 0 & 0 \\ 0 & -2H & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & 8H-k_3 \end{pmatrix}, \\
&\textit{Case VI. } S = \begin{pmatrix} -2H & 1 & 0 & 0 \\ 0 & -2H & 0 & 0 \\ 0 & 0 & 4H & \beta_1 \\ 0 & 0 & -\beta_1 & 4H \end{pmatrix}
\end{aligned} \tag{4.7}$$

for some smooth functions k_2, k_3, k_4, ν. In Cases I and III, the induced metric $g_{ij} = g(e_i, e_j) = \langle e_i, e_j \rangle$ of M is given by $g_{ij} = \varepsilon_i \delta_{ij} \in \{-1, 1\}$, in Cases II and IV, it is given by

$$g = \begin{pmatrix} \varepsilon_1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -\varepsilon_1 \end{pmatrix}$$

for $\varepsilon = \pm 1$, whereas in Cases V and VI, it takes the form

$$g = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

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